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**VARIATIONAL PRINCIPLE FOR PENETRATOR
DYNAMICS USING BILINEAR FUNCTIONAL
AND ADJOINT FORMULATION**

C. N. SHEN

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INTRODUCTION

The elastic-plastic stress-strain relations for a rod have been derived by T. Wright (ref 1), and the differential equations of the rod itself have been cast into variational form. The variational principle using bilinear functional and adjoint formulation has served as a basis to determine numerical solutions by the finite element method. This principle has now been extended to coupling systems such as in the impact dynamics. The present hyperbolic type partial differential equation has two dependent and two independent variables, with coupling in the spatial domain. This new formulation is also ready to be used for the coupled impact problem which is given in the Appendix of this report.

THE VARIATIONAL PRINCIPLE

A dynamical system can be modeled by the matrix vector partial differential equation.

$$L(\zeta) y(\zeta) = -Q(\zeta) \quad (1)$$

with appropriate boundary and initial conditions. In the above equation, L is a matrix linear operator in both spatial and temporal domain, y is a vector dependent variable, Q is a vector forcing function, and ζ represents all independent variables, both spatial and temporal.

The inner product $\langle \rangle$ of an adjoint forcing function \bar{Q} and the solution $(y(\zeta))$ of Eq. (1) can be used for the purpose of estimation. This inner product is $\langle Q, y \rangle$.

¹Wright, Thomas, W., "Nonlinear Waves in Rods," Technical Report ARBRL-TR-02324, U.S. Army Ballistic Research Laboratory, Aberdeen, MD 21005, May 1981.

An accurate estimation can be made by constructing a variational principle (refs 2-5). By using the adjoint variable y as a Lagrange multiplier for Eq. (1) and adding to $\langle Q, y \rangle$, we have

$$J_1[y, \bar{y}] = \langle \bar{Q}, y \rangle + \langle \bar{y}, (Q + Ly) \rangle = \langle \bar{Q}, y \rangle + \langle \bar{y}, Q \rangle + \langle \bar{y}, Ly \rangle \quad (2)$$

To keep the system symmetrical, let us define the adjoint system as

$$\bar{L}(\xi) \bar{y}(\xi) = -\bar{Q}(\xi) \quad (3)$$

By using the original variable y as a Lagrange multiplier for Eq. (3) and adding to $\langle Q, y \rangle$, we have

$$J_2[y, \bar{y}] = \langle Q, \bar{y} \rangle + \langle y, (\bar{Q} + \bar{L}\bar{y}) \rangle = \langle Q, \bar{y} \rangle + \langle y, \bar{Q} \rangle + \langle y, \bar{L}\bar{y} \rangle \quad (4)$$

The relationship of the adjoint system to the original system is

$$D = \langle y, Ly \rangle - \langle y, \bar{L}\bar{y} \rangle = D_e \quad (5)$$

where D is the bilinear concomitant (ref 2). Combining Eqs. (2), (4), and (5) one obtains

$$J_1 = J_2 + D_e \quad (6a)$$

In order to keep the functional symmetrical, we have

$$J_1 = \frac{\Delta}{2} [J_1 + J_2 + D_e] \quad (6b)$$

²Stacey, Weston, M., Jr., Variational Methods in Nuclear Reactor Physics, Academic Press, 1974.

³Shen, C. N. and Wu, Julian J., "A New Variational Method for Initial Value Problems, Using Piecewise Hermite Polynomial Spline Functions," ARO Report 81-3, Proceedings of the 1981 Army Numerical Analysis and Computers Conference, 1981.

⁴Shen, C. N., "Variational Principle for Gun Dynamics With Adjoint Variable Formulation," Proceedings of the Third US Army Symposium on Gun Dynamics, Volume II, May 1982, p. IV-108.

⁵Shen, C. N., "On the Extremum of Bilinear Functional for Hyperbolic Type Partial Differential Equations," ARDC Technical Report ARLCB-TR-84008, Benet Weapons Laboratory, Watervliet, NY, April 1984.

which is of the form

$$J_1 = \langle \bar{Q}, y \rangle + \langle \bar{y}, Q \rangle + \frac{1}{2} \langle \bar{y}, Ly \rangle + \frac{1}{2} \langle y, \bar{L}y \rangle + \frac{D_e}{2} \quad (6c)$$

Similarly,

$$J_2 = \langle \bar{Q}, y \rangle + \langle \bar{y}, Q \rangle + \frac{1}{2} \langle \bar{y}, Ly \rangle + \frac{1}{2} \langle y, \bar{L}y \rangle - \frac{D_e}{2} \quad (6d)$$

INTEGRAL OF BILINEAR EXPRESSION

The integral of a bilinear expression for a two-dimensional problem having a system of second order partial derivatives in time and in space can be written as

$$I = \int_{x_0}^{x_b} \int_{t_0}^{t_b} \Omega[\bar{y}(x,t), y(x,t)] dt dx \quad (7)$$

where $\Omega[\bar{y}, y]$ is a given bilinear expression in the form

$$\Omega[\bar{y}, y] = \bar{y}_t^T B y_t - \bar{y}_x^T A y_x - \bar{y}^T P y + \bar{y}_x^T \Gamma y + \bar{y}^T N y_x \quad (8)$$

The subscripts t and x indicate the partial derivatives for the functions y and \bar{y} . The matrices A, B, and P are diagonal and Γ and N are off-diagonal.

$$A = \begin{bmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1^2 & 0 \\ 0 & b_2^2 \end{bmatrix}, \quad \text{and } P = \begin{bmatrix} p_1^2 & 0 \\ 0 & p_2^2 \end{bmatrix} \quad (9)$$

$$\Gamma = \begin{bmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & \eta_1 \\ \eta_2 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{and } \bar{y} = \begin{bmatrix} \bar{w} \\ \bar{u} \end{bmatrix} \quad (10)$$

Equation (8) can be integrated by parts. Two different forms of integration and end conditions are given. The first form of the integral is obtained by integrating by parts on the adjoint variable.

$$I_a = - \iint \{ \bar{y}^T B y_{tt} - \bar{y}^T A y_{xx} + \bar{y}^T P y + \bar{y}^T P y_x - \bar{y}^T N y_x \} dt dx \\ + \int \bar{y}^T B y_t dx - \int \bar{y}^T A y_{xx} dt + \int \bar{y}^T P y dt \quad (11)$$

which gives

$$I_a = \iint -w(b_1^2 w_{tt} - a_1^2 w_{xx} + p_1^2 w + \gamma_1 u_x - \eta_1 u_x) dt dx \\ + \iint -u(b_2^2 u_{tt} - a_2^2 u_{xx} + p_2^2 u + \gamma_2 w_x - \eta_2 w_x) dx dt \\ + \int_{x_0}^{x_b} [\bar{w} b_1^2 w_t + \bar{u} b_2^2 u_t]_{t_0}^{t_b} dx + \\ \int_{x_0}^{x_b} [w a_1^2 w_x + u a_2^2 u_x]_{x_0}^{x_b} dt + \int_{t_0}^{t_b} [\bar{w} \gamma_1 u + \bar{u} \gamma_2 w]_{x_0}^{x_b} dt \quad (12)$$

$$I_a = -\langle \bar{y}, L y \rangle + \int_{x_0}^{x_b} [b_1^2 \bar{w} w_t + b_2^2 \bar{u} u_t]_{t_0}^{t_b} dx \\ - \int_{t_0}^{t_b} [a_1^2 \bar{w} w_x + a_2^2 \bar{u} u_x]_{x_0}^{x_b} dt + \int_{t_0}^{t_b} [\gamma_1 \bar{w} u + \gamma_2 \bar{u} w]_{x_0}^{x_b} dt \quad (13)$$

On the other hand, we can perform integration on the original variable to give

$$I_b = - \iint \{ y^T B \bar{y}_{tt} - y^T A \bar{y}_{xx} + y^T P \bar{y} - y^T P \bar{y}_x + y^T N \bar{y}_x \} dt dx \\ \int y^T B \bar{y}_t dx - \int y^T A \bar{y}_{xx} dt + \int y^T N \bar{y}_x dt \quad (14)$$

which gives

$$I_b = \iint (-w)(b_1^2 \bar{w}_{tt} - a_1^2 \bar{w}_{xx} + p_1^2 \bar{w} - \gamma_2 u_x + \eta_2 u_x) dt dx \\ + \iint (-u)(b_2^2 \bar{u}_{tt} - a_2^2 \bar{u}_{xx} + p_2^2 \bar{u} - \gamma_1 w_x + \eta_1 w_x) dt dx \\ + \int_{x_0}^{x_b} [w b_1^2 \bar{w}_t + u b_2^2 \bar{u}_t]_{t_0}^{t_b} dx - \int_{t_0}^{t_b} [w a_1^2 \bar{w}_x + u a_2^2 \bar{u}_x]_{x_0}^{x_b} dt + \int_{t_0}^{t_b} [w \eta_2 u + u \eta_1 w]_{x_0}^{x_b} dt \quad (15)$$

$$I_b = -\langle y, L \bar{y} \rangle + \int_{x_0}^{x_b} [b_1^2 \bar{w} w_t + b_2^2 \bar{u} u_t]_{t_0}^{t_b} dx - \int_{t_0}^{t_b} [a_1^2 \bar{w} w_x + a_2^2 \bar{u} u_x]_{x_0}^{x_b} dt + \\ + \int_{t_0}^{t_b} [\eta_2 \bar{w} u + \eta_1 \bar{u} w]_{x_0}^{x_b} dt \quad (16)$$

To keep the form symmetrical, we take the average of the previous two expressions

$$I = \frac{1}{2} I_a + \frac{1}{2} I_b = - \int_{x_0}^{x_b} \int_{t_0}^{t_b} \frac{1}{2} (\bar{y}Ly + \bar{y}Ly) dt dx + \\ + \int (\bar{y}^T B y_t + \bar{y}^T B y_t) \Big|_{t_0}^{t_b} dx - \frac{1}{2} \int (\bar{y}^T A y_x + \bar{y}^T A y_x) \Big|_{x_0}^{x_b} dt + \frac{1}{2} \int (\bar{y}^T P y + \bar{y}^T N^T y) dt \quad (17)$$

which gives

$$I = - \frac{1}{2} \langle \bar{y}, Ly \rangle - \frac{1}{2} \langle \bar{y}, Ly \rangle + \frac{1}{2} \int_{x_0}^{x_b} [b_1^2 (\bar{w}w_t + \bar{w}w_t) + b_2^2 (\bar{u}u_t + \bar{u}u_t)] \Big|_{t_0}^{t_b} dx \\ - \frac{1}{2} \int_{t_0}^{t_b} [a_1^2 (\bar{w}w_x + \bar{w}w_x) + b_2^2 (\bar{u}u_x + \bar{u}u_x)] \Big|_{x_0}^{x_b} dt \\ + \frac{1}{2} \int_{t_0}^{t_b} [(\gamma_1 + \eta_1) \bar{w}u + (\gamma_2 + \eta_2) \bar{u}w] \Big|_{x_0}^{x_b} dt \quad (18)$$

where

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \quad \bar{L} = \begin{bmatrix} \bar{L}_{11} & \bar{L}_{12} \\ \bar{L}_{21} & \bar{L}_{22} \end{bmatrix} \quad (19)$$

$$L_{11} = \bar{L}_{11} = b_1^2 \frac{\partial^2}{\partial t^2} + p_1^2 - a_1^2 \frac{\partial^2}{\partial x^2}, \quad L_{22} = \bar{L}_{22} = b_2^2 \frac{\partial^2}{\partial t^2} + p_2^2 - a_2^2 \frac{\partial^2}{\partial x^2} \quad (20)$$

$$L_{12} = \bar{L}_{12} = - \frac{\partial}{\partial x}, \quad L_{21} = \bar{L}_{21} = \frac{\partial}{\partial x} \quad (21)$$

It is noted that we have used the following relationship in obtaining Eq.

(21)

$$\eta_1 - \gamma_1 = 1 \quad \text{and} \quad \eta_2 - \gamma_2 = -1 \quad (22)$$

by comparing Eqs. (10), (21), and the last equation in the Appendix.

For a spatial and temporal partial system, Eq. (5) becomes

$$D = \int_{x_0}^{x_b} \int_{t_0}^{t_b} \bar{y}Ly dt dx - \int_{x_0}^{x_b} \int_{t_0}^{t_b} \bar{y}Ly dt dx \quad (23)$$

By equating Eqs. (11) and (14) and solving for D in Eq. (23), we are converting the double integral into single integrals in terms of the boundary conditions.

We can express the quantity D_e as the sum of three parts for end conditions D_1 , D_2 , and D_3 as

$$D_e = D_1 + D_2 + D_3 \quad (24)$$

The terms in D_1 involve the initial conditions of y and \bar{y} as

$$\begin{aligned} D_1 &= \int_{x_0}^{x_b} (\bar{y}^T B y_t - y^T B \bar{y}_t) \Big|_{t_0}^{t_b} dx \\ &= \int_{x_0}^{x_b} [b_1^2 (\bar{w} w_t - w \bar{w}_t) + b_2^2 (\bar{u} u_t - u \bar{u}_t)] \Big|_{t_0}^{t_b} dx \end{aligned} \quad (25)$$

The terms in D_2 involve the boundary conditions from the second partials of y and \bar{y} as

$$\begin{aligned} D_2 &= - \int_{t_0}^{t_b} (\bar{y}^T A y_x - y^T A \bar{y}_x) \Big|_{x_0}^{x_b} dt \\ &= - \int_{t_0}^{t_b} [a_1^2 (\bar{w} w_x - w \bar{w}_x) + a_2^2 (\bar{u} u_x - u \bar{u}_x)] \Big|_{x_0}^{x_b} dt \end{aligned} \quad (26)$$

The terms in D_3 involve the boundary conditions from the coupling terms.

$$D_3 = \int_{t_0}^{t_b} (\bar{y}^T \Gamma y - y^T \Gamma \bar{y}) dt = \int_{t_0}^{t_b} [(\gamma_1 - \eta_1) \bar{w} u + (\gamma_2 - \eta_2) u \bar{w}] \Big|_{x_0}^{x_b} dt \quad (27)$$

THE SYMMETRICAL ADJOINT SYSTEM

The adjoint independent variable τ in Figure 1 can be expressed as

$$\frac{\tau_b - \tau}{\tau_b - \tau_0} = \frac{t - t_0}{t_b - t_0} \quad (28)$$

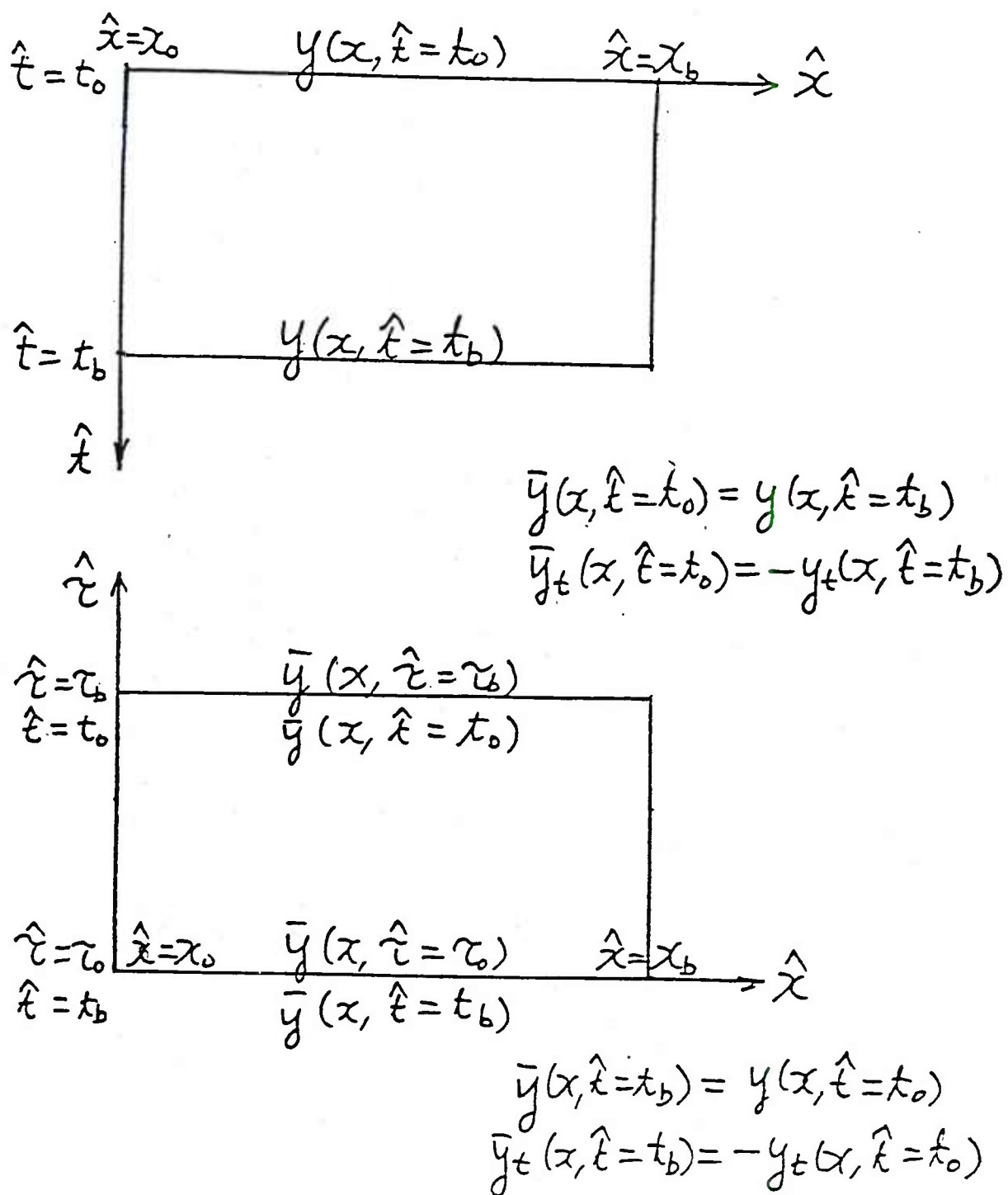


Figure 1. Image Reflection of the Adjoint System.

which gives

$$\tau = \tau_b \text{ for } t = t_o \quad (29a)$$

and

$$\tau = \tau_o \text{ for } t = t_b \quad (29b)$$

It is noted from Eq. (28) that

$$\tau_b - \tau_o = \tau_b - t_o \quad (30a)$$

$$\tau = \tau_b + t_o - t \quad (30b)$$

$$d\tau = -dt \quad (30c)$$

$$\frac{d}{d\tau} = \frac{d}{dt} \quad (30d)$$

and

$$\bar{y}(x, t) = y(x, \tau = \tau_b + t_o - t) \quad (30e)$$

Let us assume that the adjoint system shown in Figure 1 gives

$$\bar{y}(x, t=t) = y(x, t=t_b+t_o-t) \quad (31a)$$

$$\bar{y}_t(x, t=t) = -y_t(x, t=t_b+t_o-t) \quad (31b)$$

$$\bar{y}_x(x, t=t) = y_x(x, t=t_b+t_o-t) \quad (31c)$$

where t is a dummy variable for t .

We may define the adjoint system as the image reflection in the time domain of the original system. Equation (31) yields the following known initial conditions:

$$\bar{y}(x, t=t_b) = y(x, t=t_o) \quad (\text{known}) \quad (32a)$$

$$\bar{y}_t(x, t=t_b) = -y_t(x, t=t_o) \quad (\text{known}) \quad (32b)$$

The interpretation of the above equations gives the initial conditions of the original system as the far end conditions for the adjoint system, since the adjoint system is a reflected mirror of the original system in time.

INITIAL CONDITIONS FOR THE ADJOINT SYSTEM

We take a symmetrical approach for the initial conditions of the adjoint system as

$$\bar{y}(x, t=t_b) = y(x, t=t_0) \quad , \quad \bar{y}_t(x, t=t_b) = -y_t(x, t=t_0) \quad (33)$$

$$\bar{y}(x, t=t_0) = y(x, t=t_b) \quad , \quad \bar{y}_t(x, t=t_0) = -y_t(x, t=t_b) \quad (34)$$

where y and \bar{y} are given in Eq. (10). Thus Eq. (25) becomes

$$\begin{aligned} D_1 = & \int_{x_0}^{x_b} b_1^2 dx \{ [w(x, t=t_0)w_t(x, t=t_b) + w(x, t=t_b)w_t(x, t=t_0)] \\ & - [w(x, t=t_b)w_t(x, t=t_0) + w(x, t=t_0)w_t(x, t=t_b)] \} \\ & + \int_{x_0}^{x_b} b_2^2 dx \{ [u(x, t=t_0)u_t(x, t=t_b) + u(x, t=t_b)u_t(x, t=t_0)] \\ & - [u(x, t=t_b)u_t(x, t=t_0) + u(x, t=t_0)u_t(x, t=t_b)] \} = 0 \end{aligned} \quad (35)$$

Since the integrand of Eq. (35) is zero, the above satisfies Eq. (25).

THE GENERALIZED BOUNDARY CONDITIONS

Let us consider the operator L in Eqs. (19) through (21). It is assumed that elastic springs are installed at the ends such that

$$y_x(x_b, t) = K_b y(x_b, t) \quad , \quad \bar{y}_x(x_b, t) = K_b \bar{y}(x_b, t) \quad (36a)$$

$$y_x(x_0, t) = -K_0 y(x_0, t) \quad , \quad \bar{y}_x(x_0, t) = -K_0 \bar{y}(x_0, t) \quad (36b)$$

where K_b , K_0 are diagonal matrices. If Eq. (36) is substituted into Eq. (26), we have

$$D_2 = 0 \quad (37)$$

Since $D_1 = D_2 = 0$, Eq. (24) becomes D_3 as given in Eq. (27)

$$D_e = D_3 \quad (38)$$

CONDITIONS FOR THE COUPLING TERMS

The sum of the functionals $J_1 + I$ is obtained by adding Eqs. (6c) and (18) as

$$J_1 + I + \int_{x_0}^{x_b} \int_{t_0}^{t_b} (\bar{Q}y + y\bar{Q}) dx dt + T + B + V + \frac{1}{2} D_3 \quad (39)$$

where

$$T = \frac{1}{2} \int_{x_0}^{x_b} [b_1^2 (\bar{w}_t \bar{w} + \bar{w}_t w) + b_2^2 (\bar{u}_t \bar{u} + \bar{u}_t u)]_{t_0}^{t_b} dx \quad (40)$$

$$B = -\frac{1}{2} \int_{t_0}^{t_b} [a_1^2 (\bar{w}_x \bar{w} + \bar{w}_x w) + a_2^2 (\bar{u}_x \bar{u} + \bar{u}_x u)]_{x_0}^{x_b} dt \quad (41)$$

$$V = \frac{1}{2} \int_{t_0}^{t_b} [(\gamma_1 + \eta_1) \bar{u} \bar{w} + (\gamma_2 \eta_2) \bar{w} u]_{x_0}^{x_b} dt \quad (42)$$

and

$$D_3 = \frac{1}{2} \int_{t_0}^{t_b} [(\gamma_1 - \eta_1) \bar{u} \bar{w} + (\gamma_2 - \eta_2) \bar{w} u]_{x_0}^{x_b} dt \quad (43)$$

From the last two equations, one obtains

$$V + \frac{1}{2} D_3 = \int_{t_0}^{t_b} [(\gamma_1 \bar{u} w + \gamma_2 \bar{w} u)]_{x_0}^{x_b} dt \quad (44)$$

We can let Eq. (44) vanish by choosing

$$\gamma_1 = \gamma_2 = 0 \quad (45)$$

which gives

$$V + \frac{1}{2} D_3 = 0 \quad (46)$$

From Eq. (22) one obtains

$$\eta_1 = 1 + \gamma_1 = 1 \quad (46a)$$

$$\eta_2 = 1 + \gamma_2 = -1 \quad (46b)$$

Thus, the functional for the original variables and adjoint variations becomes

$$J_1 = -I + \langle \bar{Q}, y \rangle + \langle Q, \bar{y} \rangle + T + B \quad (47)$$

The sum of the two functionals $J_2 + I$ is obtained by adding Eqs. (6d) and (18) as

$$J_2 + I = \int_{x_0}^{x_b} \int_{t_0}^{t_b} (\bar{Q}y + y\bar{Q}) dx dt + T + B + V - \frac{1}{2} D_3 \quad (48)$$

where T , B , V , and $(1/2)D_3$ are given in Eqs. (40) through (43). By

subtracting $(1/2)D_3$ from V we have

$$V - \frac{1}{2} D_3 = \int_{t_0}^{t_b} [(\eta_1 u \bar{w} + \eta_2 w \bar{u})]_{x_0}^{x_b} dt \quad (49)$$

In this case we let

$$\eta_1 = \eta_2 = 0 \quad (50)$$

Then from Eq. (22) one obtains

$$\gamma_1 = -1 + \eta_1 = -1 \quad (51)$$

$$\gamma_2 = 1 + \eta_2 = 1 \quad (52)$$

Thus, the functional for the adjoint variables and original variations becomes

$$J_2 = -I + \langle \bar{Q}, y \rangle + \langle Q, \bar{y} \rangle + T + W \quad (53)$$

which gives the same form as J_1 shown in Eq. (47).

THE FIRST VARIATION

By taking the variations $\delta \bar{y}$ and δy separately, we let

$$\delta J = \delta J(\delta \bar{y}) + \delta J_2(\delta y) = 0 + 0 \quad (54)$$

Then one obtains from Eqs. (40), (41), and (47) that

$$\delta J_1(\delta \bar{y}) = -\delta I(\delta \bar{y}) + \iint Q \delta \bar{y} dx dt + \delta T(\delta \bar{y}) + \delta B(\delta \bar{y}) = 0 \quad (55)$$

where

$$\delta T(\delta \bar{y}) = \frac{1}{2} \int_{x_0}^{x_b} [b_1^2 (w_t \delta \bar{w} + w \delta \bar{w}_t) + b_2^2 (u_t \delta \bar{u} + u \delta \bar{u}_t)]_{t_0}^{t_b} dx \quad (56)$$

$$\delta B(\delta y) = -\frac{1}{2} \int_{t_0}^{t_b} [a_1^2(\bar{w}_x \delta \bar{w} + \bar{w} \delta \bar{w}_x) + a_2^2(\bar{u}_x \delta \bar{u} + \bar{u} \delta \bar{u}_x)]_{x_0}^{x_b} dt \quad (57)$$

and $-\delta I(\delta \bar{y})$ can be derived from Eq. (18) with $\gamma_1 = \gamma_2 = 0$ and $\eta_1 = \eta_2 = 1$

$$\begin{aligned} -\delta I(\delta \bar{y}) = & -\int_{x_0}^{x_b} \int_{t_0}^{t_b} \{ [b_1^2 \bar{w}_t \delta \bar{w}_t - p_1^2 \bar{w} \delta \bar{w} - a_1^2 \bar{w}_x \delta \bar{w}_x] + \\ & [b_2^2 \bar{u}_t \delta \bar{u}_t - p_2^2 \bar{u} \delta \bar{u} - a_2^2 \bar{u}_x \delta \bar{u}_x] + [\bar{u}_x \delta \bar{w} - \bar{w}_x \delta \bar{u}] \} dt dx \end{aligned} \quad (58)$$

The second term on the right side of Eq. (54) is

$$\delta J_2(\delta y) = -\delta I(\delta y) + \iint \bar{Q} \delta y \, dx dt + \delta T(\delta y) + \delta B(\delta y) = 0 \quad (59)$$

where

$$\delta T(\delta y) = \frac{1}{2} \int_{x_0}^{x_b} [b_1^2(\bar{w} \delta \bar{w}_t + \bar{w}_t \delta \bar{w}) + b_2^2(\bar{u} \delta \bar{u}_t + \bar{u}_t \delta \bar{u})]_{t_0}^{t_b} dx \quad (60)$$

$$\delta B(\delta y) = -\frac{1}{2} \int_{t_0}^{t_b} [a_1^2(\bar{w} \delta \bar{w}_x + \bar{w}_x \delta \bar{w}) + a_2^2(\bar{u} \delta \bar{u}_x + \bar{u}_x \delta \bar{u})]_{x_0}^{x_b} dt \quad (61)$$

and $\delta I(\delta y)$ can be derived from Eq. (18) with $\eta_1 = \eta_2 = 0$ and $-\gamma_1 = \gamma_2 = 1$.

$$\begin{aligned} -\delta I(\delta y) = & -\int_{x_0}^{x_b} \int_{t_0}^{t_b} \{ [b_1^2 \bar{w}_t \delta \bar{w}_t - p_1^2 \bar{w} \delta \bar{w} - a_1^2 \bar{w}_x \delta \bar{w}_x] + \\ & [b_2^2 \bar{u}_t \delta \bar{u}_t - p_2^2 \bar{u} \delta \bar{u} - a_2^2 \bar{u}_x \delta \bar{u}_x] + [-\bar{w}_x \delta \bar{u} + \bar{u}_x \delta \bar{w}] \} dt dx \end{aligned} \quad (62)$$

The adjoint equation has the same form of the original equation by dropping and adding the bars simultaneously on every variable.

Equations (55) through (58) are the key equations to be used for the finite element method. It is noted that the first variation $\delta J_1(\delta y)$ is the same as the first variation $\delta J_2(\delta y)$ by adding or dropping the bar on top of the variables and their variations. We do not need to solve for the adjoint system in Eqs. (39) through (41) since these give exactly the same solutions as the ones of the original system.

CONCLUSIONS

The functional in bilinear matrix vector form is symmetrical about the original variables and the adjoint variables. The Euler-Lagrange equations for the coupling systems are derived using the fundamental lemma of the calculus of variations. By integrating the bilinear matrix vector expression by parts, one can obtain the bilinear concomitant in terms of initial and boundary terms. The adjoint system can be arranged in a manner that it is a reflected mirror of the original system in time. Thus the initial conditions for the bilinear concomitant become zero. Generalized boundary conditions using many types of "springs" relating the various spatial partial derivatives can be defined to satisfy the boundaries of the concomitant. Algorithms are developed for use in the finite element method by taking the first variations of the functional. These algorithms are simplified because the adjoint system gives exactly the same solutions as those of the original system.

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APPENDIX

The wave equation in rods derived by T. W. Wright (ref A-1) is given as the following system:

$$w_{\xi\xi} + \frac{2\lambda}{\lambda+2\mu} u_{\xi} = \frac{c^2}{c_1^2} w_{\tau\tau} \quad (A-1a)$$

$$u_{\xi\xi} - \left[\delta \frac{\lambda+\mu}{\mu} u + 4 \frac{\lambda}{\mu} w_{\xi} \right] = \frac{c^2}{c_2^2} u_{\tau\tau} \quad (A-1b)$$

which can be transformed to

$$\left(\frac{\lambda+2\mu}{2\lambda} w_{tt} - \frac{\lambda+2\mu}{2\lambda} w_{xx} \right) - u_x = 0 \quad (A-2a)$$

$$w_x + \left(\frac{\mu}{4\lambda} \frac{c^2}{c_2^2} u_{tt} - \frac{\mu}{4\lambda} u_{xx} + \frac{2(\lambda+\mu)}{\lambda} u \right) = 0 \quad (A-2b)$$

With appropriate group of parameters, we have the following form:

$$b_1^2 w_{tt} - a_1^2 w_{xx} + p_1^2 w - u_x = 0 \quad (A-3a)$$

$$w_x + b_2^2 u_{tt} - a_2^2 u_{xx} + p_2^2 u = 0 \quad (A-3b)$$

The above system of equations can be expressed by a matrix vector form of equations as

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (A-4)$$

where

$$L_{11} = b_1^2 \frac{\partial^2}{\partial t^2} - a_1^2 \frac{\partial^2}{\partial x^2} + p_1^2, \quad L_{12} = - \frac{\partial}{\partial x} \quad (A-5)$$

A-1 Wright, Thomas W., "Nonlinear Waves in Rods," Technical Report ARBRL-TR-02324, U.S. Army Ballistic Research Laboratory, Aberdeen, MD 21005, May 1981.

$$L_{21} = \frac{\partial}{\partial x} \quad \text{and} \quad L_{22} = b_2^2 \frac{\partial^2}{\partial t^2} - a_2^2 \frac{\partial^2}{\partial x^2} + p_2^2 \quad (\text{A-6})$$

which can be written as Eq. (1) in the text.

The notations for the wave equation in rods are:

w = axial displacement

u = radial strain

$\xi = z/a$ = nondimensional axial coordinates

$\tau = ct/a$ = nondimensional time

a = elastic stored energy per unit length

$c_1 = \sqrt{(\lambda+2\mu)/\rho}$ = longitudinal wave speed

$c_2 = \sqrt{\mu/\rho}$ = shear wave speed

λ and μ are Lamé constants

The above system of equations was first developed in different form by Mindlin and Herrmann (ref A-2) and can be grouped into a single equation as:

$$\left[\left(\frac{\partial^2}{\partial \xi^2} - \frac{c^2}{c_1^2} \frac{\partial^2}{\partial \tau^2} \right) \left(\frac{\partial^2}{\partial \xi^2} - \frac{c^2}{c_2^2} \frac{\partial^2}{\partial \tau^2} \right) - \delta \frac{\lambda+\mu}{\mu} \frac{c_b^2}{c_1^2} \left(\frac{\partial^2}{\partial \xi^2} - \frac{c^2}{c_b^2} \frac{\partial^2}{\partial \tau^2} \right) \right] (w \text{ or } u) = 0 \quad (\text{A-7})$$

A-2 Mindlin, R. D. and Herrmann, G., "A One-Dimensional Theory of Compressional Waves in an Elastic Rod," Proceedings of the First U.S. National Congress of Applied Mechanics, 1950, pp. 187-191.

APPENDIX REFERENCES

- A-1. Wright, Thomas W., "Nonlinear Waves in Rods," Technical Report ARBRL-TR-02324, U.S. Army Ballistic Research Laboratory, Aberdeen, MD 21005, May 1981.
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